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Uncertainty Principle for Generalized Diffusions

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It is shown that with any generalized diffusion a complementary variable, the *probabilistic group velocity* (PGV), can be associated, such that the uncertainties in the position (diffusion) and its complementary PGV (like momentum) satisfy a Heisenberg-type uncertainty relation. It is shown that the bound is achieved in the linear Gaussian case. In the statistical steady state, the PGV vanishes identically. The uncertainty in the PGV is an achievable upper bound to the rate of the RMS value of the diffusion. The PGV is further related to the entropy rate of the diffusion process.

1. INTRODUCTION

The similarity between the free Schrödinger equation and the diffusion equation has been the motivation for the search for a stochastic interpretation of quantum mechanics. Such a program is carried out in stochastic mechanics (Blanchard *et al.*, 1987; Carlen, 1988), where it is also shown that a stochastic analog exists for the Heisenberg uncertainty relations. Fürth derived an uncertainty relation for the (pure) diffusion equation. It bounds the product of the uncertainties in position and what he called the *osmotic* velocity (Fürth, 1933). The notion of osmotic velocity is generalized to more general semi-martingales (Blanchard *et al.*, 1987), using the forward and backward Kolmogorov equations. However, it does not satisfy a Heisenberg-type uncertainty relation. In this paper, we show that with any arbitrary continuous semi-martingale we can associate a complementary variable, the *probabilistic group velocity* (PGV), such that the product in the uncertainties in the semi-martingale (the position) and its complementary PGV (like momentum) satisfies an uncertainty relation. The next section defines this

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PGV, and the uncertainty principle it satisfies is proven. In Section 3, we show that the minimum uncertainty is achieved in the linear Gaussian case. Interpretations and properties of both the PGV and the uncertainty are then discussed in Section 4. Finally, Section 5 connects the entropy as function of time to the PGV. This paper is an extended version of a conference paper (Verriest and Shin, 1991), and emphasizes a particle density interpretation.

2. PGV AND UNCERTAINTY PRINCIPLE

Let $(\Omega, \mathscr{F}, P)$ be a complete probability space, and let $w(t)$ be the standard Wiener process defined on it. Let $f(x, t)$ and $g(x, t)$ be Lipshitz continuous functions $\mathbf{R} \times \mathbf{R}_+ \to \mathbf{R}$, so that the semi-martingale $x(t)$ described by the Itô differential equation

$$dx = f(x, t)\, dt + g(x, t)\, dw(t) \tag{1}$$

is well defined and has continuous sample paths. Let further x_0 be the initial condition, which we take to be random (measurable with respect to $\mathscr{F}$). Let also its distribution be absolutely continuous with respect to Lebesgue measure, so that a density ρ_0 exists. It is well known that an equivalent description of the Itô system is given by the conditional probability density $\rho(x, t \mid y, s)$ of the solution $x(t)$ of the Itô equation (1), starting in y at time $s < t$. This conditional density satisfies the Fokker–Planck (or forward Kolmogorov) equation on $\mathbf{R}$

$$\frac{\partial \rho}{\partial t} = -\frac{\partial (f\rho)}{\partial x} + \frac{1}{2}\frac{\partial^2 (g^2 \rho)}{\partial x^2} \tag{2}$$

with initial condition

$$\lim_{t \downarrow s} \rho(x, t \mid y, s) = \delta(x - y) \tag{3}$$

A *particle interpretation* may be given for this. Its position is modeled by the semi-martingale $x(t)$. The Itô equation is a nonlinear Ornstein–Uhlenbeck equation in a field with *drift* $f(x, t)$ and *diffusion* $g(x, t)^2$. Let its initial position ($s = 0$) be chosen at random with density $\rho_0(x)$. Denote the probability that the particle will be found at time t in an infinitesimal interval of length dx near x by $\rho(x, t)\, dx$. It is easily shown that $\rho(x, t)$ also satisfies the same Fokker–Planck equation (2) on $\mathbf{R}$, but with initial condition $\rho(x, 0) = \rho_0(x)$.

Alternatively, one can consider an ensemble of particles, all obeying the dynamics (1), but driven by independent Brownian motions $\{w_i(t) : i \in \mathscr{I}\}$, where $\mathscr{I}$ is some index set, which is not necessarily countable. If the position of the particle indexed by i at time t is $x_i(t)$, then $\rho(x, t)$ is the *particle density* near x at time t, i.e.,

$$\rho(x, t)\,dx = \mathscr{N} \sum_{\mathscr{I}} \mathbf{1}_{[x, x+dx)}(x_i(t)) \tag{4}$$

The $\mathscr{N}$ is a normalization factor. The independence of the driving terms guarantees that the individual particles only react to the drift and diffusion fields, and not with each other. They are in effect "test particles." The Fokker–Planck equation is then interpreted as a *diffusion equation*. The latter model will be utilized to develop the concept of the uncertainty principle.

Define now a quantity

$$u(x, t) = f(x, t) - \frac{1}{2\rho(x, t)} \frac{\partial}{\partial x} [g^2(x, t)\rho(x, t)] \tag{5}$$

which can be interpreted as a velocity, given by a deterministic drift term $f(x, t)$, and a diffusion-related term, an interaction with the density field $\rho(x, t)$. For a constant diffusion, $g^2(x, t) = \sigma^2$, the negative of the second term in (5) is referred to as the osmotic velocity in Blanchard *et al.* (1987, p. 43). It will be shown below that the PGV defined by us, unlike the osmotic velocity, satisfies an uncertainty principle. Note also that (5) can be expressed alternatively by

$$u(x, t) = f(x, t) - \tfrac{1}{2} g^2(x, t) \frac{\partial}{\partial x} \log[g^2(x, t)\rho(x, t)] \tag{6}$$

The diffusion equation (2) can be written as a *continuity* equation

$$\frac{\partial \rho}{\partial t} + \frac{\partial j}{\partial x} = 0 \tag{7}$$

where we introduced the new quantity

$$j = \rho u \tag{8}$$

This j can be interpreted as a probability current (flux), so that the $u(x, t)$ defined in (5) is a velocity term, which we shall call the *probabilistic group velocity* (PGV). The name reflects that it depends on the particle density, and is therefore not an attribute of an individual particle in the ensemble $\mathscr{I}$. Equation (8) shows that with the ensemble-of-particles interpretation, the PGV is the net average velocity of the group of particles crossing through position x at time t; i.e., *if all particles were moving uniformly*. Indeed the momentum carried through x is $j(x, t)\,dx = [\rho(x, t)\,dx]u(x, t)$. If this is distributed uniformly over the probability mass $\rho(x, t)\,dx$, then obviously $u(x, t)$ is the equivalent group velocity.

Alternatively, consider at time t the interval with boundaries a and b. Let this interval be displaced, compatibly with the velocity field defined by the PGV. Thus at an infinitesimal amount of time dt later new boundaries are respectively $a+u(a, t)\,dt$ and $b+u(b, t)\,dt$. The probability mass in the original interval is $\int_a^b \rho(x, t)\,dx$, while in the displaced one (at time $t+dt$) it is

$$\int_{a+u(a,t)\,dt}^{b+u(b,t)\,dt} \rho(x, t+dt)\,dx$$

The difference between the two is, up to first order

$$\begin{aligned}
\Delta\rho(a, b, t) &= \int_{a+u(a,t)\,dt}^{b+u(b,t)\,dt} \left[\rho(x, t)+\frac{\partial\rho(x, t)}{\partial t}\,dt\right] dx - \int_a^b \rho(x, t)\,dx \\
&= \int_{a+u(a,t)\,dt}^{a} \left[\rho(x, t)+\frac{\partial\rho(x, t)}{\partial t}\,dt\right] dx + \int_a^b \left[\frac{\partial\rho(x, t)}{\partial t}\,dt\right] dx \\
&\quad + \int_b^{b+u(b,t)\,dt} \left[\rho(x, t)+\frac{\partial\rho(x, t)}{\partial t}\,dt\right] dx \\
&= -\rho(a, t)u(a, t)\,dt - \int_a^b \frac{\partial(\rho(x, t)u(x, t))}{\partial x}\,dx\,dt \\
&\quad + \rho(b, t)u(b, t)\,dt \\
&= [\rho(b, t)u(b, t)-\rho(a, t)u(a, t)]\,dt - \int_a^b d[\rho(x, t)u(x, t)]\,dt \\
&= 0.
\end{aligned} \tag{9}$$

Hence *the probability mass inside a domain which moves with the (nonuniform) velocity field u remains constant.* With the particle representation, this means that the number of particles in the comoving domain remains constant. However, this does not mean that the individual Brownian particles cannot move in and out of the boundary. It just says that the *net* number remains constant, in equilibrium, relative to the motion according to the PGV.

Example. Of particular interest is the case $f(x, t)\equiv 0$ and $g(x, t)\equiv 1$ (the pure constant diffusion), for which the PGV is

$$u(x, t) = -\frac{1}{2t}\frac{\partial\rho(x, t)}{\partial x}$$

Using the Green's function, we can express this in terms of the initial density $\rho_0(x)$ as

$$u(x, t) = \frac{1}{2t}\left[x - \frac{\int_{-\infty}^{\infty} y\rho_0(y)\exp[-(y-x)^2/2t]\,dy}{\int_{-\infty}^{\infty} \rho_0(y)\exp[-(y-x)^2/2t]\,dy}\right]$$

If the initial position is deterministic and zero, we have the standard Wiener process (Brownian motion). The density $\rho(x, t)$ is then normally distributed with mean zero and variance t, so that

$$u(x, t) = \frac{x}{2t}$$

Consider now a spatial average

$$\langle u\rangle_D(t) = \int_D \rho(x, t)u(x, t)\,dx$$

of this group velocity $u(x)$ over an interval $D=(a, b)$, which may be all of $\mathbf{R}$, but is such that $g^2(b, t)\rho(b, t) = g^2(a, t)\rho(a, t)$. From (5) one easily gets

$$\langle u\rangle_D(t) = \langle f\rangle_D(t) \tag{10}$$

We introduce the (local on D) deviation in the PGV

$$\begin{aligned}\tilde{u}_D(x, t) &= u(x, t) - \langle u\rangle_D(t)\\ &= \tilde{f}(x, t) - \frac{1}{2\rho(x, t)}\frac{\partial}{\partial x}[g(x, t)^2\rho(x, t)]\end{aligned} \tag{11}$$

where $\tilde{f}(x, t) = f(x, t) - \langle f\rangle_D(t)$. For notational simplicity, the subscript D will be omitted when the domain is understood.

The variance Δu^2 is defined as $\langle\tilde{u}^2\rangle$, and is a measure for the uncertainty in the PGV. It is readily obtained (suppressing the time coordinate) as

$$\Delta u^2 = \Delta f^2 + \left\langle g^2\frac{\partial f}{\partial x}\right\rangle + \frac{1}{4}\int\frac{1}{\rho}\left(\frac{\partial g^2\rho}{\partial x}\right)^2 dx \tag{12}$$

It should be emphasized again that this is an ensemble property of the group of particles near x. It is the local (in interval D) spatial fluctuation in the group velocity PGV at time t.

On the other hand, the uncertainty in the position given by Δx^2 has a single-particle interpretation: It is the variance in the position at time t, i.e., $x(t)$, of a test particle, starting at time 0, with a randomly chosen initial position with density $\rho_0(\cdot)$. The following uncertainty principle gives a lower bound to the product of these uncertainties.

Theorem 1. Uncertainty Principle. Consider an ensemble of independent particles obeying the semi-martingale dynamics (1), with initial density $\rho_0(x)$. Let $\rho(x, t)$ denote the particle density at time t. Let $D=(a, b)$ be an interval on the real line $\mathbf{R}$ for which

$$g(a, t)^2\rho(a, t)=g(b, t)^2\rho(b, t) \tag{13}$$

$$(a-\langle x\rangle_D)g(a, t)^2\rho(a, t)=(b-\langle x\rangle_D)g(b, t)^2\rho(b, t) \tag{14}$$

Then the uncertainties in the position $x(t)$ and its associated PGV $u(t)$ in D obey the uncertainty principle:

$$\Delta x\,\Delta u\geq\tfrac{1}{2}|\langle g^2+2(x-\langle x\rangle)(f-\langle f\rangle)\rangle| \tag{15}$$

Proof. Consider for an arbitrary constant C the obvious inequality

$$\left(\tilde{u}(x)+\frac{x-\langle x\rangle}{C}\right)^2\geq 0 \tag{16}$$

Multiplying (16) by ρ and integrating over the interval D, one gets precisely, after a partial integration,

$$\Delta u^2+2\frac{\langle(x-\langle x\rangle)(f-\langle f\rangle)\rangle}{C}$$

$$-\frac{[(x-\langle x\rangle)g^2(x)\rho(x)]|_{\partial D}}{C}$$

$$+\frac{\langle g^2\rangle}{C}+\frac{\Delta x^2}{C^2}\geq 0$$

where ∂D denotes the boundary (the points a and b of D). Hence, by the assumption on the interval D, we get the inequality

$$\Delta u^2+2\frac{\langle(x-\langle x\rangle)(f-\langle f\rangle)\rangle}{C}+\frac{\langle g^2\rangle}{C}+\frac{\Delta x^2}{C^2}\geq 0 \tag{17}$$

Since C is arbitrary, it can now be chosen optimally to minimize the left-hand side of (17). Let thus

$$C=-\frac{\Delta x^2}{\frac{1}{2}\langle g^2\rangle+\langle(x-\langle x\rangle)(f-\langle f\rangle)\rangle} \tag{18}$$

and substitution in the inequality (17) yields finally the uncertainty relation (15), which is valid for all time. ■

The theorem says that the product of the uncertainties in the position of the particle (whose motion is modeled by a diffusion or continuous semimartingale) and the uncertainty in the PGV is bounded below by the right-hand side of (15). For the standard Wiener process considered in the example above, the bound given by the uncertainty principle is $\frac{1}{2}$. Noting that $\Delta u = 1/(2\sqrt{t})$ and $\Delta x = \sqrt{t}$, it is seen that the equality holds in the uncertainty principle. This is not coincidental, as shown next.

3. LINEAR GAUSSIAN PROCESSES

The example considered in Section 2 generalizes easily to the time-dependent linear system

$$dx = a(t)x\,dt + b(t)\,dw \tag{19}$$

For a Gaussian initial condition, the distribution of $x(t)$ remains Gaussian, with mean $m(t)$ and variance $P(t)$ satisfying, respectively,

$$\dot{m}(t) = a(t)m(t) \tag{20}$$

$$\dot{P}(t) = 2a(t)P(t) + b(t)^2 \tag{21}$$

The probabilistic group velocity can be computed explicitly. Indeed, since

$$\rho(x, t) = \frac{1}{[2\pi P(t)]^{1/2}} \exp\left\{-\frac{[x - m(t)]^2}{2P(t)}\right\} \tag{22}$$

we find

$$u(x, t) = a(t)x + \frac{b(t)^2}{2}\,\frac{x - m(t)}{P(t)} \tag{23}$$

Averaging over the entire space $D = \mathbf{R}$, we get

$$\begin{aligned}\langle u\rangle(t) &= \int \left[a(t)x + \frac{b(t)^2}{2}\,\frac{x - m(t)}{P(t)}\right]\rho(x, t)\,dx \\ &= a(t)m(t) \end{aligned} \tag{24}$$

from which the variance is easily obtained,

$$\Delta u^2 = \int \left[a(t) + \frac{b(t)^2}{2P(t)}\right]^2 [x - m(t)]^2 \rho(x, t)\,dx \tag{25}$$

$$= \left[a(t) + \frac{b(t)^2}{2P(t)}\right]^2 P(t) \tag{26}$$

Substitution in the left-hand side of the uncertainty principle shows the equality of both sides in inequality (15). Thus we have proved the following:

Corollary 1. Equality in the uncertainty principle is achieved for the *linear Gaussian* case.

As a final remark, note that for a time-invariant stable system, a is negative, and a steady-state variance P_∞ exists. It is given by

$$P_\infty = \frac{b^2}{2|a|} \tag{27}$$

It follows from

$$\tilde{u}(x, t) = a\frac{P(t) - P_\infty}{P(t)}[x - m(t)] \tag{28}$$

that if $P(t) < P_\infty$, then $\operatorname{sgn}(\tilde{u}) = \operatorname{sgn}[x - m(t)]$. That is, a particle at position x has a tendency to *escape* away from the mean [so that $P(t)$ increases], while if $P(t) > P_\infty$, then $\operatorname{sgn}(\tilde{u}) = \operatorname{sgn}[x - m(t)]$, and the tendency is to *regulate* toward the mean. These tendencies have to be understood in a statistical sense. For the statistical steady state, the right-hand side of the equality (28) is zero, so that the spatial average of the fluctuation in the group velocity satisfies then $\Delta u = 0$. In fact, more can be said:

Corollary 2. A time-invariant linear Gaussian system with $a < 0$ and $b \neq 0$ is in the *statistical steady state* if and only if $u(x, t) \equiv 0$ for all x and t.

Proof. It follows readily from (23) that for a and b as given, $u(x, t) \equiv 0$ if and only if $m = 0$ and $2aP + b^2 = 0$, i.e., if the statistical steady state is reached. ■

For every particle escaping a neighborhood near x, another one enters on average. A statistical steady state exists also in the time-variant case if $a(t)$ is proportional (with a constant positive factor) to $-b(t)^2$.

4. STATIONARITY AND PGV UNCERTAINTY

The characterization of stationarity in the linear Gaussian case is so nice that one may wonder if it generalizes to the nonlinear time-invariant case. If the lower bound in the uncertainty principle is nonzero, then $u(x, t)$ cannot be constant! We will show in this section that in fact the lower bound in the uncertainty principle is zero in the *statistical steady state* (SSS). Then we shall move on to interpret the PGV uncertainty as a bound on the rate of the RMS value of x.

While an SSS may exist for a time-dependent system, we shall here uniquely look at the time-invariant equation (1) with drift $f(x)$ and diffusion $g(x)$. It is clear that this time invariance does not necessarily imply that the solution $\rho(x, t)$ to (2) is independent of t. However, if an SSS exists, it is characterized by a solution $\rho(x)$ to the time independent equation (a second-order ODE)

$$[f(x)\rho(x)]' - \tfrac{1}{2}[g(x)^2\rho(x)]'' = 0 \tag{29}$$

with of course the constraint that $\int_{\mathbf{R}} \rho(x)\, dx = 1$. It is readily seen that the *unique* (by virtue of the assumed Lipshitz conditions) solution to (29) is given by

$$\rho(x) = \frac{N_0}{g(x)^2} \exp\left[2 \int_{x_0}^{x} \frac{f(z)}{g(z)^2}\, dz\right] \tag{30}$$

If this function is integrable, it is a potential steady-state density. The N_0 is then the appropriate normalization factor.

If the system is *stochastically stable*, then it can be shown that

$$\lim_{t \to \infty} \rho(x, t) = \rho_\infty(x) \tag{31}$$

with $\rho_\infty(x)$ given by (30). General necessary and sufficient conditions for stochastic stability, i.e., condition (31), are not known, but sufficient conditions are derivable, for instance, via the geometric theory for parabolic equations (Hale *et al.*, 1984).

Theorem 2. The statistical steady state, if it exists, is characterized by the PGV being identically zero, i.e.,

$$\forall x, \forall t \geq 0: \quad u(x, t) \equiv 0 \Leftrightarrow \text{SSS}$$

Proof. (i) Sufficiency: If $u(x, t) \equiv 0$, then obviously $\partial(\rho(x, t)u(x, t))/\partial x \equiv 0$, hence, it follows from (2) that $\partial\rho/\partial t \equiv 0$, implying stationarity.

(ii) Necessity: If $\rho(x, t)$ is independent of t, say $\rho(x, t) = \rho_\infty(x)$, then the definition of the PGV (5) implies that for a time-invariant system (1), u is not a function of t. But by the Fokker–Planck equation (2), $d\rho(x)u(x)/dx = 0$. Hence $\rho(x)u(x) = A$, where A is some constant. Since for a steady state $\rho(x) \to 0$ as $|x| \to \infty$, then either $u(x) \to \infty$ if $A \neq 0$, or $u(x) \equiv 0$.

Consider now

$$\int_{\mathbf{R}} A\,dx = \int_{\mathbf{R}} [u(x)\rho(x)]\,dx \tag{32}$$

$$= \int_{\mathbf{R}} \left[f(x)\rho(x) - \frac{1}{2}\frac{d(g(x)^2\rho(x))}{dx} \right] dx \tag{33}$$

$$= \langle f \rangle - \int_{\mathbf{R}} d[g(x)^2\rho(x)] \tag{34}$$

But in the statistical steady state the average of x is constant. From (1) this implies $0 = d\langle x \rangle / dt = \langle f \rangle$. The second term in (34) is also zero by virtue of the assumption (14). It follows thus that the constant A can only be zero, thus proving that $u(x) \equiv 0$. ■

It follows from this theorem that if there exists a statistical steady state for a time-invariant system, then in this SSS there is no uncertainty principle, i.e., the lower bound of the uncertainty principle is zero, since $u \equiv 0$ implies $\Delta u = 0$. If a stochastic system approaches the SSS, then somehow the PGV should approach zero. The following result relates the PGV with the rate of the RMS value of the diffusion in the general time-varying case.

Theorem 3. The rate of change in the RMS value of the position of the Brownian particle obeying the dynamics (1) is upper bounded by the PGV uncertainty.

Proof. Returning to the original (time-dependent) Itô equation, it follows from Itô differential rule that the variance P obeys

$$dP = \langle g^2 + 2(x - \langle x \rangle)(f - \langle f \rangle) \rangle\,dt \tag{35}$$

Upon substitution in the uncertainty relation, we have

$$\Delta u\,\Delta x \geq \frac{1}{2}\left|\frac{dP(t)}{dt}\right| \tag{36}$$

but, since also $P = \Delta x^2$, this gives the equivalent

$$\left|\frac{d\Delta x}{dt}\right| \leq \Delta u \tag{37}$$

which states that Δu is bounded below by the maximally (achievable) rate of growth of the RMS value for the process $x(t)$. ■

Finally we derive a class of systems for which u is constant. If $u(x, t) = u_0$, then equations (7) and (8) lead to the first-order PDE

$$u_0 \frac{\partial \rho}{\partial x} + \frac{\partial \rho}{\partial t} = 0 \tag{38}$$

which has the general solution $F(x - u_0 t)$. If an initial density $\rho_0(x)$ is given, the solution for the density is

$$\rho(x, t) = \rho_0(x - u_0 t) \tag{39}$$

The initial density moves *uniformly* with velocity u_0. Changing to a comoving coordinate $\xi = x - u_0 t$, we find that then $\rho_0(\xi)$ is a steady-state distribution for a system with drift $f(\xi) - u_0$ and diffusion $g(\xi)^2$, or, in the original spatial coordinate, $f(x - u_0 t)$ and $g(x - u_0 t)^2$, respectively.

5. ENTROPY AND PGV

In this section, we give another relation where the group velocity u plays a significant role. Let x_t be the solution to the Itô equation at a specific time t, and let $\rho_t(x)$ be its density function. The (differential) entropy is defined as

$$H(x_t) = -\int_{-\infty}^{\infty} \rho_t(x) \log \rho_t(x)\, dx \tag{40}$$

It is customary to write x in the argument of the entropy to denote the random variable whose entropy is computed. This is, however, an abuse of notation, as the entropy H in (40) is not a function of x. Now consider this entropy as parametrized by time t. The entropy rate can then be defined as the derivative of (40) with respect to t, provided this exists. Similar to the development in Costa and Cover (1984), we relate the entropy rate for the PGV.

Theorem 4. The entropy rate of the diffusion equals the average *divergence* of the PGV, i.e.,

$$\frac{dH(x_t)}{dt} = \left\langle \frac{\partial u}{\partial x} \right\rangle \tag{41}$$

Proof. The infinitesimal interval $[x, x + dx)$ contributes an amount

$$h(x, t)\, dx = -\rho(x, t) \log \rho(x, t)\, dx$$

to the entropy (40). Making use of equations (7) and (8), one gets

$$
\begin{aligned}
\frac{\partial h}{\partial t}\,dx &= -\frac{\partial \rho}{\partial t}\log \rho\, dx - \rho\,\frac{\partial \log \rho}{\partial t}\,dx \\
&= -\frac{\partial \rho}{\partial t}\,(1+\log \rho)\, dx \\
&= \frac{\partial (u\rho)}{\partial x}\,(1+\log \rho)\, dx \\
&= \frac{\partial u}{\partial x}\,\rho(1+\log \rho)\, dx + u(1+\log \rho)\, d\rho \\
&= \frac{\partial u}{\partial x}\,\rho\, dx + d[u\rho \log \rho]
\end{aligned} \tag{42}
$$

Integration over x yields then the statement (41). ■

By the integrability of $\rho(x, t)$, the probability flux $j(x, t)$ must approach zero as $|x| \to \infty$. Integrating by parts, we can therefore rewrite equation (41) as

$$
\frac{dH(x_t)}{dt} + \int_{-\infty}^{\infty} u(x, t)\, d\rho(x, t) = 0 \tag{43}
$$

If one interprets the infinitesimal form (42) as

$$
\frac{h(x, t+dt)\, dx - h(x, t)\, dx}{[\rho(x, t)\, dx]\, dt} = \frac{\partial u}{\partial x} + \frac{d[u\rho \log \rho]}{\rho\, dx} \tag{44}
$$

then the left-hand side shows a change in the entropy contributed by the infinitesimal interval $[x, x+dx)$ per unit probability mass, or in the particle interpretation, *per particle* in $[x, x+dx)$. The right-hand side shows that this comes from two effects: one is the local (near x) change in u, the other is some "fluctuation" [the second term in (44), which is rather nondescriptive]. The nice property is that the *net* contribution of this second term to the overall entropy rate is zero. Hence, one could call the divergence part $\partial u/\partial x$ the *effective entropy* (denoted by h_{eff}) contributed per particle at x in a time unit.

Similarly, the infinitesimal form of equation (43) yields

$$
\frac{\partial h}{\partial t}\,dx = -u\,\frac{\partial \rho}{\partial x}\,dx + d[\,j \log \rho] + dj \tag{45}
$$

Only the first term $-u(\partial\rho/\partial x)\,dx$ contributes effectively to the total entropy. So

$$u(x, t) = -\frac{h_{\text{eff}}(x, t+dt)\,dx - h_{\text{eff}}(x, t)\,dx}{[\partial\rho(x, t)\,dx/\partial x]\,dt} \tag{46}$$

Thus, the change in the effective entropy is due to the gradient in ρ. Per time unit and per unit of "gradient in ρ," this change in the effective contribution by the infinitesimal interval near x to the entropy is exactly the PGV u.

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